# Hopf-Galois module structure of degree $p$ extensions of $p$-adic fields

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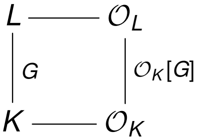


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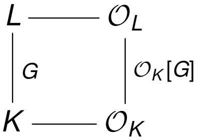


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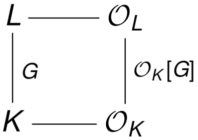


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The study of this question is closely related with the **ramification** of $L/K$.

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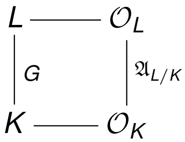


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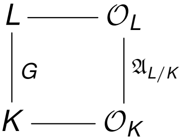


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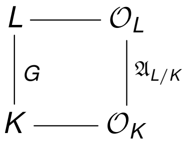


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$$1 \leq t \leq \frac{pe}{p-1}.$$

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- If $t \geq \frac{pe}{p-1} - 1$ ($L/K$ is almost maximally ramified), $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if $n \leq 4$, where

$$\frac{t}{p} = [a_0; a_1, \dots, a_n].$$

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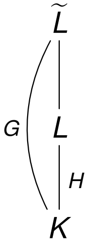

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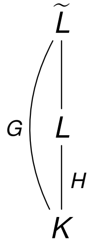


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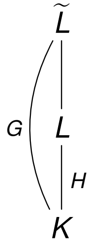


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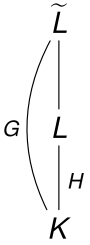


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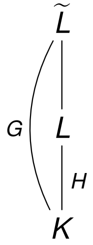


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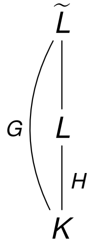


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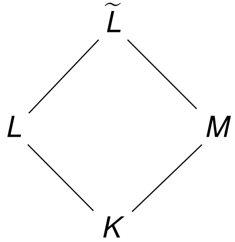
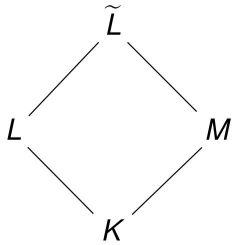
We shall characterize the $\mathfrak{A}_{L/K}$-freeness of $\mathcal{O}_L$.




$\widetilde{L}/K$ has a unique quadratic subextension $M/K$. The extension $\widetilde{L}/M$ **is cyclic of degree** $p$.

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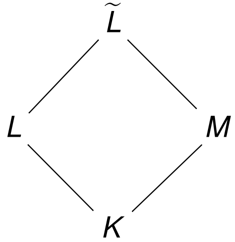


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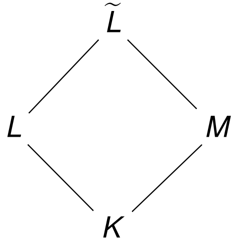


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### Corollary

*If $\widetilde{L}/K$ is not totally ramified, then $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if $\mathcal{O}_{\widetilde{L}}$ is $\mathfrak{A}_{\widetilde{L}/M}$-free.*

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We claim:

- If $a = 0$, then $t = \frac{2pe}{p-1}$ and $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.
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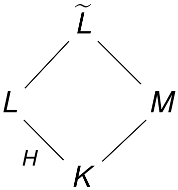


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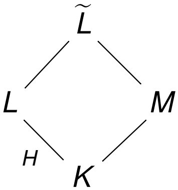


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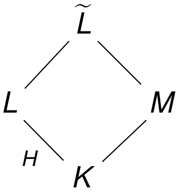


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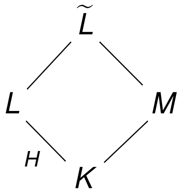

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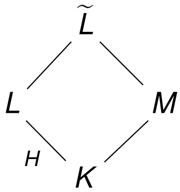


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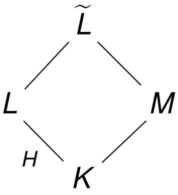


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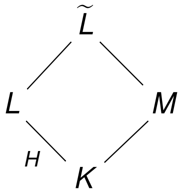


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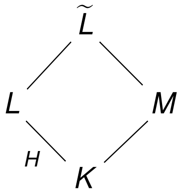


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Let us define $\alpha = \gamma\tau(\gamma)$. Then:

- $v_L(\alpha) = 1$ and $\alpha^p \in \mathcal{O}_K$.
- $w \cdot \alpha^j = \lambda_j \alpha^j$ for every $0 \leq j \leq p-1$.
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We find $\{v_i\}_{i=0}^{p-1}$ $\mathcal{O}_K$-basis of $\mathfrak{A}_{L/K}$ such that

$$v_i \cdot \alpha^j = \delta_{ij} \alpha^j.$$

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$\Longrightarrow \mathfrak{A}_{L/K}$ is the maximal $\mathcal{O}_K$-order in $H$ and $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.

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Write $\ell = pa_0 + a$ and denote $\theta := \pi_L^a$.

# Basis of $\mathfrak{A}_\theta$

## Proposition

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$$\{\pi_K^{-\nu_i} w^i\}_{i=0}^{p-1},$$

*where for each $0 \leq i \leq p-1$,*

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This is a direct consequence from the fact that

$$v_L(w^i \cdot \theta) = a + i\ell, \quad 0 \leq i \leq p-1.$$

# Basis for the associated order

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$$n_i = \min_{0 \leq j \leq p-1-i}(\nu_{i+j} - \nu_j).$$

Note that $n_i \leq \nu_i$ for every $0 \leq i \leq p-1$. The equalities hold if and only if $\mathfrak{A}_\theta = \mathfrak{A}_{L/K}$.

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Then,

$$(k+3)\frac{a}{p} - (k+1)\frac{a}{p} = \frac{2a}{p} > 1 \implies \nu_{k+2} - \nu_k \geq 1$$

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$$\mathfrak{c} = pe - \frac{p-1}{2}t = \frac{p-1}{2}\left(\frac{2pe}{p-1} - t\right) = p\left(e + \frac{p-1}{2} - \nu_{p-1}\right) + a.$$

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Weak condition: $\mathfrak{c} \geq a$.
Strong condition: $\mathfrak{c} > a$.

**Corollary**

*If $a \mid p-1$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free. If additionally $t < \frac{2pe}{p-1} - 2$, then the converse holds.*

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Let us assume that $n \geq 3$.

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For $\theta = \pi_L^a$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if $\mathfrak{A}_\theta$ is $\mathfrak{A}_{L/K}$-principal.

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This is closely related with the set

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$$E = \Big\{ a'_{2i+2} q_{2i+1} + q_{2i} \,|\, 0 \leq i < \frac{n-1}{2}, \\ 0 \leq a'_{2i+2} \leq a_{2i+2} \text{ or } a_{2i+2} - 1 \Big\}.$$

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If $n = 3$, $E = \{q_0, q_1 + q_0, \ldots, (a_2 - 1) q_1 + q_0, q_2\}$.

The $\mu_{j,i}^{(k)}$ are defined by

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Hence,

$$v_K(\mu_{j,i}^{(k)}) = v_K(C(w^j, w^{k+i})) + \nu_j - \nu_k - n_i,$$

where $C(w^j, w^{k+i})$ is the coefficient of $w^j$ in the expression of $w^{k+i}$.

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(1) If $j \neq k + i$, then $\overline{\mu}_{j,i}^{(k)} = 0$.

(2) If $h \notin E$,

$$\overline{\mu}_{k+i,i}^{(k)} = \begin{cases} 1 & \text{if } h = p \\ 1 & \text{if } \widehat{(k+1)\frac{a}{p}} < \widehat{h\frac{a}{p}} \\ 0 & \text{otherwise} \end{cases}$$

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Assume that $k + i > p - 1$ and let $m = k + i - (p - 1)$.

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Call $d_m = \max\{d \in \mathbb{Z} \mid d \text{ even}, \nu_{m+d} = \nu_m + \frac{d}{2}\}$.

(1) If $j \not\equiv 2 \pmod{m}$, then $\overline{\mu}_{j,i}^{(k)} = 0$.

(2) If $m < p-1$ and $j < m$ or $j > m + d_m$, then $\overline{\mu}_{j,i}^{(k)} = 0$.

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$\deg(P) \leq p-2 \Longrightarrow \overline{\det(M(\alpha))} \neq 0$ for some $u$.

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Consider the $i$-th column of $M(\alpha)$, where $h = p - i$ takes the values

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$$\implies \det(M(\alpha)) \equiv 0 \pmod{\mathfrak{p}_K}.$$

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*Let $M/K$ be a ramified quadratic extension of p-adic fields with $\xi_p \in M - K$ and assume that*

(a) $t = \frac{2pe}{p-1}$.

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Given some ramification parameters, we may choose $L/K$.

**Example**

Assume that $p > 3$. There is some degree $p$ extension $L/K$ such that $\widetilde{L}/K$ is weakly ramified and $\mathcal{O}_L$ is not $\mathfrak{A}_{L/K}$-free.

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There is some degree $p$ extension $L/K$ of $p$-adic fields such that $\mathcal{O}_{\widetilde{L}}$ is $\mathfrak{A}_{\widetilde{L}/M}$-free but $\mathcal{O}_L$ is not $\mathfrak{A}_{L/K}$-free.

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# Thank you for your attention